

# Index

- $T(\cdot)$ , 113  
 $\vee$ , 26  
 $\wedge$ , 26  
 $\neg$ , 26  
 $\rightarrow$ , 26  
 $H_n$ , 65, 120  
 $O(\cdot)$ , 119  
 $R(\cdot, \cdot)$ , Ramsey number, 157  
 $[n]$ , 188, 206  
 $\Delta$ , 167, 173  
 $\mathbb{N}$ , *see* natural numbers, 91  
 $\Omega(\cdot)$ , 119  
 $\Omega$ , sample space, 216  
 $\mathbb{Q}$ , *see* rational numbers  
 $\mathbb{R}$ , *see* real numbers  
 $\Sigma$ , 3, 113  
 $\Theta(\cdot)$ , 119  
 $\mathbb{Z}$ , *see* integers  
 $\lceil \cdot \rceil$ , 94  
 $\chi$ , chromatic number, 167, 172  
 $\binom{n}{k_1, k_2, \dots, k_r}$ , multinomial, 197  
 $\binom{n}{k}$ , binomial coefficient, 182  
     formula, 187  
     parity, 192  
     recursion, 182  
 $\mathcal{A}_{\text{ODD}}$ , 104–105  
 $\emptyset$ , *see* empty set  
 $\exists$ , 32  
 $\forall$ , 31  
 $\gamma \approx 0.577$ , 120  
 $\gcd(\cdot, \cdot)$ , 130–132  
 $\text{lcm}(\cdot, \cdot)$ , 141  
 $\omega(\cdot)$ , 119  
 $\mathcal{P}$ , *see* power set  
 $\prod$ , 3  
 $\text{rem}(\cdot, \cdot)$ , 129  
 $\{\cdot\}$  (rounding operation), 83  
 $\Delta$ , deviation, 310  
 $\sqrt{2}$ , 20  
 $e \approx 2.71828$ , 69  
 $e^{i\theta}$ , 127  
 $o(\cdot)$ , 119  
 1-to-1, 184, 327  
 2-COLORING, 428  
 2-contact EBOLA, 11  
 2-SAT, 427  
 3-SAT, 422, 426, 429  
     derandomize, 308  
     exact version, 429  
     probabilistic method, 308  
     randomized algorithm, 430  
 3-sigma rule, 315  
 $3n + 1$  problem, *see* Collatz problem  
  
*A Mathematician's Apology*, 129  
 ABC News, Primetime Live, 161  
 Abel, 126  
 absolute deviation, 320  
 Ackermann, Wilhelm, 334  
 acyclic, 159  
 addition rule, 114  
 adjacency list/matrix, 146, 154, 156  
 Adleman, Leonard, 136  
  
 Akra-Bazzi formula, 96  
 algebraic number, 337  
 algorithm analysis/runtime, 113  
 Alice in Wonderland, 145  
 aliens, 244  
 alphabet, 341  
 AM-GM inequality, 82, 83  
 AM-HM inequality, 175  
 anagrams, 197, 206  
 AND/ $\wedge$ , 26  
 Anna Karenina, 101  
 Anna Karenina principle, 387  
 antecedent, 41  
 Antoine de Saint-Exupery, viii  
 approximation algorithm  
     BIN-PACKING, 428  
     KNAPSACK, 428  
 Archimedes, 129  
 Aristotle, 211  
 arithmetic expression, 104  
 arithmetic mean, 82  
 arithmetic series, 59  
 ASCII code, 17  
*Ask Marilyn*, 213  
 assignment  
     animals to habitats, 177  
     exam slots, 177  
     radio frequencies, 11, 177  
 asymptotic analysis, 113–129  
     linear, 117  
 augmenting path, 170  
 AUTO-GRADE, 387, 394  
 automaton, *see* DFA  
     nondeterministic, 354  
 average, 277  
      $n$ -dice, 309  
     reciprocals, 312  
     reduces variance, 320  
 axiom, well-ordering principle, 20  
  
 $B_\infty$ , 332  
 Backus, Peter, 244  
 balanced parentheses, 101  
 BALANCED-BIPARTITE-CLIQUE, 427  
 Ballot Theorem, 239  
 balls and bins, 324  
 balls and weighings, 52  
 Banach's matchbox, 276, 303  
 barber paradox, 335  
 Barksdale, Jim, 225  
 Barton, Bernard, 281  
 barycentric coordinate, 160  
 base case, induction, 57  
 base case, recursion, 86  
 Basel problem, 126  
 batch testing, 302  
 Batman™, 200  
 Bayes' Theorem, 233, 272  
 Beck, 258  
 Bell, Eric T., 41  
 bell-curve, 318  
 Bernoulli  
     diffusion, 304  
     maximizes variance, 322  
     number of heads, 265  
     random variable, 265  
     random walk, 265  
     sum, 265, 266  
 Bernoulli numbers, 58, 70  
 Bernoulli's inequality, 64  
 Bernoulli, Jakob, 58  
 Bertrand's Postulate, 144  
 betting, horse racing, 288  
 Bezout coefficients, 140  
 Bezout's identity, 130, 135, 140, 141  
 bias, in sampling, 230  
 big-oh, omega, theta, 117–119  
 Biggs, N., viii, 145  
 bijection, 184–186, 188, 209, 327, 335  
     and shortest paths, 194  
     counting goody-bags, 185  
     counting menus, 185  
     definition, 184  
 binary strings  
     counting, 207  
     finite, 336  
     infinite, 334  
     programs, 334  
 binomial, 187  
     coefficient, 67, 84, 182, 187, 267  
         central, 67  
         generalized, 193  
         identities, 192  
         Pascal's identity, 98  
         sum of squares, 194  
     properties, 274  
     sums, 192  
 Binomial distribution, 240, 241, 266  
     expected value, 294  
     maximum probability, 274  
     numerically computing, 274  
     random walk, 273  
 Binomial Theorem, 67, 70, 127, 138, 142, 187, 208  
     proof by induction, 192  
 BIN-PACKING, 428  
     3/2-approximation, 428  
     2-approximation, 428  
     inapproximability, 428  
 bins with capacities, 210  
 BIPARTITE, 427  
 bipartite graph, 161–166  
     adjacency matrix, 170  
     chromatic number, 168  
     complete, 164, 171  
     König-Egerváry Theorem, 172  
     no odd-cycle, 170  
 bipartite matching, 162  
     Hall's condition, 163–164  
 birthday paradox, 246–248, 255  
 birthday puzzle, 171  
 BITWISE-OR, 271  
 Bletchly Park, 136  
 Bombieri, Enrico, 25  
 Boole's inequality, 218  
 Boole, George, 211

- Boolean games, 430  
 BOUNDED- $k$ , 426  
 Bouton, Charles, 82  
 breadth first search (BFS), 158  
 breathalyzer, 237  
 Breathnach, Sarah, 243  
 Brouwer Fixed Point Theorem, 160  
 Buffett, Warren, 215  
 build-up counting, 182–184, 186  
      $\binom{n}{k}$  recursion, 182  
     binary sequences,  $k$  ones, 182  
     goddy-bags, 183  
     induction, 183  
     sequences with constraints, 183  
 build-up expectation, 297  
 build-up probability, 241  
 Busy Beaver, non-computable, 402  
 butane, 92  
  
 Caldwell, Sarah, 293  
 Cantor diagonalization, 332  
 Cantor set, 338  
 Cantor's Theorem, 337  
 Cantor's transfinite realm, 333  
 Cantor, Georg, 7, 327, 337  
 Cantor-Bernstein Theorem, 328  
 capture-recapture, 276  
 cardinality, 328  
     irrational numbers, 337  
     power set, 337  
     square,  $\mathbb{R}^2$ , 337  
 Carleson, Lennart, 179  
 Carroll, Lewis, 145  
 Cartesian product, 23  
 Catalan number, 112, 195  
 Catalan recursion, 98  
 Cauchy, Augustin-Louis, 82  
 Cauchy-Schwarz Inequality, 321  
 Cayley's formula, 196  
 Cayley, Arthur, 92  
 CDF, 264–265  
     Binomial, 270  
     from PDF, 269  
     uniform, 270  
     waiting time, 270  
 ceiling function, 94  
 Central Limit Theorem, 318  
 certificate/certifier, 414  
 CFG, 365–377  
      $\mathcal{L}_{\text{bal}}$ , 367  
     CFG<sub>bal</sub>, 366  
     arithmetic, 369  
     complement, 373  
     concatenation, 368  
     context sensitive, 376  
     derivation, 366  
         leftmost, rightmost, 375  
     English, 366  
     Ginsburg-Rice Theorem, 375  
     implement DFA, 368  
     intersect with regular, 375  
     intersection, 373  
     Kleene star, 368  
     markup language, 370  
     parse tree, 369  
     production rule, 365  
     programming, 366  
     prove grammar works, 367  
     pumping, 373, 375  
     recursion, 365  
     right-linear, 375  
     terminal, 365  
     unary, 375  
     union, 368  
     variable, 365  
 CFL, *see* CFG  
 Chavez, Cesar, 3  
 Chebyshev, 144  
 Chebyshev Inequality, 316  
  
 checkers, 429  
 chess, 429  
 chessboard, 156, 159  
 Chevalier de Méré, 211  
 Chicken Nugget Theorem, 140  
 Chinese postman, 156  
 Chinese Remainder Theorem, 142  
 Chomsky, Noam, 365, 370  
     Normal Form, 369, 374, 375  
 “choose”,  $\binom{n}{k}$ , 182  
 chromatic number, 167, 173  
     bipartite graph, 168  
 Chua, Amy, 339  
 Church-Turing Thesis, 384, 403  
     extended, 406  
 Churchill, Winston, vii, 365, 387  
 circuit complexity, 420  
 circuits, 418  
     CLIQUE, 419  
     AND, 419  
     SORTER, 419  
     and Turing Machines, 418, 426  
     sorter, 425  
 CIRCUIT-SAT, 418  
 claim  
     equivalence, 28  
 clinical trial, 273  
 clique, 8, 173, 174  
     CLIQUE, 409, 411, 415, 416, 425  
     CLIQUE-VERIFY, 412  
 closure, 40, 54  
 Coeliac, 238  
 cognitive revolution, 349  
 Cohen, Paul, 334  
 coin  
     biased, 269  
     fair, 211  
     fair from biased, 232, 240  
     two-headed, 232, 237  
 coincidence, 246  
 Collatz problem, 14, 94  
 COLOR-VERIFY, 412  
 coloring, 161–168, 179, 348  
     5-color theorem, 173  
     and conflicts, 167  
     and assignment, 166  
     and independent set, 175  
     and scheduling, 166  
     chromatic number, 167  
     Four-color theorem, 158  
     greedy, 167, 172  
     planar graph, 167  
     polyhedron faces, 177  
     tree, 168, 173  
     weak 2-coloring, 175  
     with oracle, 173  
 COLORING, 415, 416, 425  
 combinations, 186, 187  
 combinatorial proofs, 193  
 common sums, 115  
 community integration, 178  
 comparison scale, 13, 52, 67  
 complement graph, 155  
 complete graph, 147  
     bipartite, 147, 164, 171  
 complex numbers, 127  
 complexity, computing problem, 343  
 component, 155  
 COMPOSITE-VERIFY, 412  
 compositions, 194  
 compressible, 402  
 compression tree, 92  
 computable function, 400, 402  
 computable number, 402  
 computation trace, 350  
 computing prob., *see* decision prob.  
     unsolvable, 385  
 computing, definition, 339–349  
 Conan Doyle, Arthur, 85, 225  
  
 concatenation, 346  
     Kleene star, 343  
     language, 342  
     positive, 347  
 conditional expectation, 281, 283  
     derandomize, 308  
     linear, 306  
     total expectation, 283  
 conditional PDF, 273  
 conditional probability, 225–243  
     definition, 225  
     inclusion-exclusion, 235  
     Monty Hall game, 227  
     pair of boys, 228  
     prosecutor's fallacy, 238  
     Simpson's paradox, 239  
     transposed conditional, 230  
     traps, 229  
 conditional variance, 323  
 conflict resolution, 167  
 CONFUSED, 387  
 conjunction, 26  
 connected, 67, 155, 158  
     strongly, 152  
 connected components, 168  
 connection puzzle, 6  
 coNP, 425  
 consequent, 41  
 constant rule, 114  
 contention resolution, 257  
 context free language, *see* CFG  
 context sensitive grammar, 376  
 continued fraction, 24, 97, 98  
 continuum hypothesis, 333  
 continuum,  $c$ , 332, 333  
     and  $\mathbb{R}^2$ , 337  
     and irrational numbers, 337  
     uncountable, 333  
 contraction (edge), 257  
 contradiction, principle of, 47  
 contrapositive, 45  
 convex hull, 160  
 Cook, Steve, 418  
 Cook-Levin Theorem, 418  
 countable, 328–332  
     algebraic numbers, 337  
     Cartesian product, 336  
     even numbers, 329  
     finite binary strings, 331  
     finite set, 328  
     integers, 329  
     listing, 329–330  
     non-negative integers, 328  
     odd numbers, 329  
     rationals, 331  
     recursive set, 338  
     square numbers, 329  
     union, 330, 336  
 counter, 394  
 counterexample, 44, 63  
 counting, 179–211  
     and bank security, 190  
     balls in bins, 210  
     bijection, 184–186, 194–196  
     bijections, 188  
     binary sequences, 179  
     build-up, 182  
     bulk pack, 179  
     cases, 180  
     chessboard squares, 192  
     combinations, 187  
     combinatorial proofs, 193  
     committees, 181  
     compositions, 194  
     digit-sum, 210  
     genotypes and phenotypes, 190  
     goody-bags, 179, 197  
     graphs, 189, 190, 210  
     inclusion-exclusion, 199

- indistinguishable castles, 186
- injections, 188
- integer solutions, 185, 202, 210
- labeled trees, 196
- license plates, 180, 188
- matched parentheses, 195
- menus, 180
- min-cuts, 258
- monotonic sequences, 191, 194
- multiples, 208
- multiplicity rule, 186
- non-attacking castles, 191
- non-empty partitions, 191
- onto functions, 202
- outcomes, 181
- parking functions, 196
- passwords, 181
- paths, 191, 208
- paths above diagonal, 191, 195
- perfect matchings, 196
- permutations, 186
- poker hands, 187, 190
- product rule, 180
- queen-king positions, 186
- race outcomes, 180
- rectangles, 194
- rooted binary trees, 195
- rooted forests, 196
- safe-key distribution, 192
- seating arrangements, 194
- sequences, 179–181, 197
- social security numbers, 189
- subsets, 181, 185
- sum rule, 179, 188
- surjections, 188
- ternary strings, 188
- Towers of Hanoi, 192
- walks, 195
- with repetition, 197
- within case, 180
- coupon collecting, 295
- covariance, 307
- craps, casino game, 233
- Crusie, Jennifer, 365
- cryptography, 133–138
  - private key, 134, 137
  - public key, 136, 137
- cumulative dist. function, *see* CDF
- cut, maximum, 176
- cut-vertex, 157
- cycle, 146, 147
  - Euler, 156
- cyclic shift, 52
  
- D'Alembert, 221
- d-PDA, det. PDA, 356, 371, 377
- Dahl, Roald, vii
- Dalio, Ray, 431
- darts, 235
- dating, 164, 165, 171, 275
  - proposer wins, 166, 172
- de Bruijn, Nicolaas, 179
- decidable, 383
  - $\mathcal{L}_{\text{ALL-DFA}}$ , 398
  - $\mathcal{L}_{\text{CFG}}$ ,  $\mathcal{L}_{\text{CFG-00}}$ , 398
  - $\mathcal{L}_{\text{DFA}}$ ,  $\mathcal{L}_{\text{DFA-00}}$ , 397
  - $\mathcal{L}_{\text{EMPTY-CFG}}$ , 398
  - $\mathcal{L}_{\text{EMPTY-DFA}}$ , 397
  - $\mathcal{L}_{\text{EQ-DFA}}$ , 398
  - $\mathcal{L}_{\text{FINITE-CFG}}$ , 398
  - $\mathcal{L}_{\text{FINITE-DFA}}$ , 398
- CFL, 396
  - closure, 400
  - regular language, 396
  - union, 388
- decision problem, 339–349
  - GRAPH-DISTANCE- $D$ , 341
  - $\overline{\text{YES}}$ -set, 341
  - and calculation, 348
  - and optimization, 340
  - and search, 348
  - coloring, 348
  - complexity, 343
  - independent set, 348
  - language, 341
  - multiple of 4, 346
  - sorted, 346
- decision tree, 92
- decision versus search, 429
- decoding, noisy channel, 273
- Dedekind, Richard, 337
- defective dollar bill, 188
- definitions, 46
- degree, 145
  - joint distribution, 276
  - weighted, 151, 156
- degree distribution, 146
- degree histogram, 147
- degree sequence, 146, 154, 155
- derandomize
  - 3-SAT, 308
  - conditional expectation, 307
  - independent set, 308
  - max-cut, 307
- derangement, 208, 299
  - build-up counting, 201
  - formula, 202
- det. finite automaton, *see* DFA
- determinant, 98, 109
- deviation from mean, 309, 310
- Dewey Defeats Truman, 230
- DFA, 345, 349–365
  - addition, 359
  - algebraic trace, 350
  - closed, regular operations, 355
  - closure, 363
  - complement, 352
  - concatenation, 353
  - convert to reg. expression, 360
  - count, 360
  - design, 350
  - equality, 360
  - external memory, 356
  - instruction set, 349
  - intersection, 352
  - isomorphism, 360
  - Kleene star, 353
  - language, 350
  - nondeterministic, 354, 362
  - probabilistic, 360
  - pumping lemma, 364
  - regular expression, 355
  - regular language, 351
  - reversal, 362, 363
  - state
    - $\overline{\text{YES}}$  OR  $\overline{\text{NO}}$ , 349
    - accept or reject, 349
  - state-diagram, 349
  - transducer, 359, 360
  - two-tape, 360
  - union, 352
  - vending machine, 351
- diagonalization, 332
- dice
  - 7-sided from 5-sided, 240
  - non-transitive, 215, 433
  - PDF of sum, 261
- Die Hard, 131, 140
- differencing operator, 110
- differentiation method, 126
- diffusion, Bernoulli, 304
- Dijkstra, Edsger, vii, 339
- Dilworth's Theorem, 153, 159
- Dirac's Theorem, 174
- direct proof, 41
- directed graph, 152, 159
  - and flow, 152
  - degree sum, 152
  - in-degree, 152
  - out-degree, 152
  - partial order, 152
- DIRECTED-HAM-PATH, 425
- discrete objects, 15–25
- disjoint paths, 178
- disjunction, 26
- disjunctive normal form, 37
- disprove implication, 44
- distinct representatives problem, 171
- division, 129
- DNF, 37
- dodecahedron, planar drawing, 174
- dominating set, 168, 173, 174
  - maximum degree  $\Delta$ , 174
  - positive degree graph, 175
  - viral marketing, 174
- domino, 156
- DOM-SET, 425, 428
- double induction, 80
- Dr. Seuss, 71
- Drake equation, 244
- draughts, 429
  
- [EASY], 168
- EBOLA, 7
- Eccles, Peter J., 55
- edge, 145
  - contraction, 257
  - crossing, 158
  - subdivide, 159
- edge-probability, 212, 213, 233
- Edison, Thomas Alva, 179
- efficiency, 403–413
- Ehrlich, Paul, 85
- Einstein, Albert, 15, 85
- empty set, 15, 16
- encoding, 384
- encryption, 134
- Enigma machine, 136, 325
- Entscheidungsproblem, 334
- enumerator, 396
- epidemics, 2-contact model, 7
- equivalence, 45
- Erasothenes, 139
- Erdős, Paul, 22, 129, 144, 157, 258
- Escher, M. C., 85
- estimate, 123
- ethane, 92
- Euclid's algorithm, 130, 134, 139
  - extended, 140
- Euclid's Lemma, 84, 133
- Euler, 126, 150, 164
- Euler cycle, 156, 168
- Euler path, 156
- Euler's Invariant, 150, 158
- Euler's totient function  $\varphi$ , 142, 202
- Euler-Mascheroni constant, 120
  - irrational, 144
- EULER-PATH, 408
- event, 217, 221
  - combining, 217
  - independent, 243–258
  - probability, 213–215
- event-probability, 214, 217, 260
- evidence, 414
- exact 3-SAT, 429
- exchange paradox, 292
- excluded middle, 21, 25
- EXCLUSIVE-OR, 223
- $E[\cdot]$ , expected value, 278
- expectation, *see* expected value, *see* expected value
  - expected value, 277–286, 289
  - absolute deviation, 320
  - Bernoulli, 279
  - Binomial, 279, 294
  - build-up, 297
  - conditional, 281, 306

- derangements, 299
- exponential, 280
- exponentiation, 306
- from CDF, 289
- from sample space, 278
- height, 281
- iterated, 295
- joint-PDF, 290
- linearity, 293
  - infinite sum, 307
- product, 297–298
- record breaking, 300
- sum, 293
- sum of dice, 278
- sum of indicators, 299
- total expectation, 283
- two coin flips, 278
- uniform, 279
- vs. typical value, 287
- waiting time, 280, 285, 294
- experiment, 211, 212, 221
- exponential, 3
- exponential time
  - 2-player games, 429
- expression tree, 92
- extended Euclid Algorithm, 140
- face, planar graph, 150
- face, polygonal, 158
- face, polyhedron, 150
- facility location, 168
- FACTOR, 425
- factorial power, 193
- factorial, integration bound, 120, 121
- factoring, 429
  - unary, 411
- failure probability, 254
- fast algorithm, polynomial, 405
- Fermat's Little Theorem, 67, 136, 138, 142
  - counting proof, 196
  - Euler's Extension, 142
- Fermat, last theorem, 53
- Fermat, Pierre de, 211
- Fermi method, 244
- Fibonacci sequence, 89, 94, 109, 123, 127, 139
  - ancestry tree, 108
  - bounds, 90
  - build-up counting, 183
  - consecutive products, 94
  - gcd, 94
  - generating function, 111
  - Sanskrit poetry, 89
  - sums, 94
- Fibonacci tiling, 100
- file compression, lossless, 209
- FIND-MINIMUM, 408
- finite automaton, *see also* DFA
- finite binary strings, countable, 331
- finite state machine, *see* DFA
- fishy, 21, 47–48
- 5-color theorem, 173
- fixed point theorem, Brouwer, 160
- flow, 152
- Floyd, Robert, 104
- FOCS-twin, 246, 255
- Ford-Fulkerson algorithm, 178
- forest, 109
- Forster, E. M., 71
- Fortnow, Lance, 413
- Four-Color Theorem, 150
- Four-color theorem, 158
- free movie ticket problem, 255
- FREQ-ITEMS, 423
- frequency, 212
- frequent items, 413
- friends paradox, 155
- friendship network, 8
- Frost, Robert, 387
- full vertex, 108
- Fundamental Theorem of Arithmetic, 75, 83, 84, 133
- Furst, Merrick, 413
- Gale-Shapley, 172
- Gale-Shapley Theorem, 165
- Galileo Galilei, 7, 41, 259, 293
- Gallup poll, 229, 230, 322
- Gallup, George, 229
- Galton's paradox, 221, 235
- gambler's ruin, 250–252
  - expected time, 290
- game theory, 292
- game tree, 92
- game, guess high or low, 266
- game, matching, 266
- Garey, 423, 427
- Gates, Bill, 215
- Gauss, 58, 142
- Gauss, Carl F., 134
- GCD, 130–132, 139, 141
  - Bezout's identity, 130
  - Euclid's algorithm, 130
  - positive linear comb., 130
  - proofs, 130
  - relatively prime, 130, 132
- gcd( $\cdot, \cdot$ ), *see also* GCD
- generating function
  - digit-sum, 210
  - gambler's ruin, 256
  - integer solutions, 210
  - solving recurrence, 111
- genetics, 190, 238, 292
- geometric distribution, 268
- geometric mean, 82
- geometric sum, 56, 59
  - from differentiation, 126
  - integration bound, 121
- gift exchange, 222
- Ginsburg, Ruth Bader, 243
- Ginsburg-Rice Theorem, 375
- go, board game, 429
- Gödel, Kurt, 327, 333, 390
- Goldbach conjecture, 144, 391
- Goldberg, Mark K., viii
- golden ratio,  $\phi$ , 90, 100
- Goldwasser, Shafi, 197
- Google™, viii
- gossip, 223, 237
- graceful labeling, 160
- graceful tree conjecture, 160
- Graham, Ronald L., viii
- Graham-Cumming, 325
- grammar, 365–377
- graph, 17–18, 145–179
  - acyclic, 173
  - affiliation, 18
  - bipartite, 161, 170
  - card, 157
  - coloring, 166
  - complement, 155
  - complete  $K_n$ , 147
  - complete bipartite  $K_{n,\ell}$ , 147
  - configuration, 159
  - conflict, 18
  - connected, 145, 173
  - counting, 190
  - cycle  $C_n$ , 147
  - deck, 157
  - degree, 145
  - degree-twins, 156, 203
  - DFA, 349
  - diameter, 156
  - directed, 152
  - edge, 145
  - hypercube, 156
  - interval, *see* interval graph
  - isomorphism, 145
  - line, *see* line graph
  - loop, 150
  - multigraph, 150
  - parallel edge, 150
  - path, 145
  - path/line  $L_n$ , 147
  - Petersen, *see* Petersen graph
  - planar, 149
  - polyhedron, 158
  - positive degree, 175
  - problem solving, 153, 159, 177
  - problems, 173
    - CLIQUE, 173
    - DOMINATINGSET, 173
    - INDEPENDENTSET, 173
    - VERTEXCOVER, 173
  - regular, 148
  - relabeling, 145
  - residual, 164
  - simple, undirected, 150
  - star  $S_{n+1}$ , 147
  - subdivision, 159
  - vertex, 145
  - weighted, 150
  - wheel  $W_{n+1}$ , 147
- graph theory, 145–179
- graphical sequence, 155
- greatest common divisor, *see* GCD
- greedy
  - $k$ -center, 176
  - coloring, 167, 172
  - matching, 171
- Grimmett, Geoffrey, viii
- growth rate, 117
  - cubic, 118
  - exponential, 118
  - factorial, 118
  - linear, 117, 118
  - log, 118
  - loglinear, 118
  - nested sum, 119
  - polynomial, 118
  - quadratic, 118
  - rules, 119
  - superlinear, 118
  - transitive, 119
- Hadamard matrix, 110
- Haldane, J. B. S., 292
- half-full vertex, 108
- Hall's Theorem, 163, 170
  - König-Egerváry Theorem, 172
  - stronger sufficient condition, 164
- Halmos, Paul, 3
- $\mathcal{L}_{\text{HALT}}$ , 392
- HALTING PROBLEM, 387, 392
- halting state, 377
- HAM-CYCLE, 426
- HAM-PATH, 409, 426
- Hamilton, William, 174
- Hamiltonian cycle, 168, 174
  - Dirac's Theorem, 174
  - Ore's Theorem, 174
- Hamiltonian path
  - tournament, 174
- HAMILTONIAN-VERIFY, 412
- Handshaking Theorem, 148
  - Sperner's Lemma, 160
- [HARD], 168
- hard problems, 14
- Hardy, Godfrey H., 21, 129
- harmonic mean, 82
- Harmonic number, 65, 120, 123
  - integration bound, 120
  - sum of, 125
- hash function, 249
- hashing, 248
  - collision, 249



- collision probability, 250
  - Havel-Hakimi, 155
  - height(), 106, 108
  - Hello World, 387, 394
  - Helly's Theorem, 66
  - Hempel's Paradox, 39
  - Hidden Markov Model, 360
  - Hilbert's 10th problem, 396
  - Hilbert, David, 327, 334, 390
  - Hipassus, 20
  - Hitler, 136
  - hitting time, 276
    - expected, 290
  - HMM, 360
  - Hoeffding bound, 318, 324
  - Hoeffding, Wassily, 318
  - Hopcroft, John E., viii
  - Horner's method, 107
  - Horning, Jim, 225
  - HTML, 371
  - Huff, Darrell, 161
  - hydrocarbon, 92
  - hydrocarbons, 92
  - hypercube, 156
  - Hypergeometric distribution, 275
  
  - I spy* game, 203
  - identically distributed, 262
  - IF AND ONLY IF, 45
  - IF... THEN...  $\rightarrow$ , 26
    - proving, 30
  - image segmentation, 178
  - implication, 26–28
    - proving, 30
  - IMPLIES, 27
  - in-degree, 152
  - inapproximability, 428
  - inclusion-exclusion, 199, 207
    - conditional probability, 235
    - counting multiples, 201
    - counting paths, 208
    - derangement, 201, 202
    - digit-sum, 207
    - Euler's totient function, 202
    - formula, 200
    - integer solutions, 202
    - onto functions, 202
    - valid permutations, 202
  - independence, 243–258
    - coin flips, 243
    - eye color, 243
    - many events, 245
    - sex, 243
    - weather, 243
  - independent set, 173, 174, 348
    - and coloring, 175
    - and vertex cover, 175
    - interval graph, 175
    - probabilistic method, 308
    - Turán's bound, 175
  - index of summation, 113
  - indicator, sum of, 299–301
  - IND-SET, 421, 423, 425
  - induced subgraph, 157
  - induction, 55–71, 85
    - 6-color theorem, 168
    - and dominos, 57
    - and well-ordering, 63, 338
    - backward, 79, 82
    - Binomial Theorem, 192
    - checklist, 78
    - climbing a ladder, 78
    - double, 80
    - exponential, 79
    - leaping, 74
    - local minimum, 82
    - on graphs, 148
    - ordinary vs. strong, 76
    - predicate, 56
    - principle of, 57
    - proof of by well-ordering, 63
    - strong, 70, 75–78
    - stronger claim, 71
    - structural, 101
    - sum of integer powers, 58
    - wrong, 59, 61, 64, 83
  - induction step, 57
  - inequality
    - AM-GM, 82, 83
    - AM-HM, 175
    - Boole's, 218
    - Lubell-Yamamoto-Meshalkin, 258
  - infinite binary strings, 334
    - $B_\infty$ , 332
    - binary expansion, 333
    - eventually constant, 337
    - eventually zero, 333
    - uncountable, 332
  - infinite loop, 383
  - infinite probability space, 219
  - Infinity, 327–339
    - algebra, 337
    - and computing, 334
  - information, 402
  - injection, 188, 327, 335
  - instruction set, 349
  - integer programming, 429
  - integers, 15
  - integration method, 119–125, 129
    - and recurrences, 121
    - factorial, 120, 121
    - geometric sums, 121
    - Harmonic number, 120
    - Stirling's formula, 120
    - sum of integer powers, 120
  - intersect CFL with regular, 375
  - interval graph, 175, 428
  - interviewing problem, 275
  - INT-PROGRAM, 429
  - inverse, 49
    - modular, 135
  - invertible, 327
  - irrational
    - $\sqrt{2}$ , 20
    - $\sqrt{3}$ , 24
  - IS-SORTER, 425
  - iso-butane, 92
  - isomorphic, 154
  - ISOMORPHIC, 415
  - isomorphism, 145, 157
    - invariant, 146
  - IS-PRIME, 415
  - iterated expectation, 295–296
  - iterated variance, 323
  
  - James, William, 113
  - Jensen's inequality, 83, 306
  - Johnson, 423, 427
  - joint probability, 231
  - joint-PDF, 273
  - Josephus problem, 40, 69, 100
  - Joyce, James, 101
  - jugs puzzle, Die Hard, 131
  
  - $\mathcal{K}$ , Kolmogorov complexity, 402
  - $k$ -center, 168
    - greedy algorithm, 176
  - $k$ -dominated, 258
  - $k$ -SAT, 425
  - König-Egerváry Theorem, 172
  - Königsberg/Kaliningrad map, 151
  - Karp, Richard M., 423
  - Kelly–Ulam conjecture, 157
  - Kempner series, 127
  - key, private, 134
  - King Bruce and the spider, 281
  - Kleene star, 343, 346, 353, 361
    - unary, 348
    - vs. wildcard symbol, 346
  - Kleene, Stephen, 327
  - Kleinberg, Jon, viii
  - KNAPSACK, 2-approximation, 428
  - knight, 156, 159
  - Knuth, Donald E., vii, viii
  - Kolmogorov complexity,  $\mathcal{K}$ , 402
    - non-computable, 402
  - Kolmogorov, Andrey, 211
  - Kotzig, 160
  - Kraft inequality, 109
  - Kuratowski's Theorem, 159
  
  - $\mathcal{L}^+$ , positive concatenations, 347
  - $L$ -tile land, 72, 81
  - LAME, 231
  - language, 339–349
    - $\mathcal{L}_{\text{bal}}$ , 342
    - $\mathcal{L}_{\text{even-pal}}$ , 342
    - $\mathcal{L}_{\text{mult}}$ , 380
    - $\mathcal{L}_{\text{punc-pal}}$ , 371
    - $\mathcal{L}(M)$ , 350
    - $\mathcal{L}_{\text{ALL-CFG}}$ , 397
    - $\mathcal{L}_{\text{ALL-DFA}}$ , 397
    - $\mathcal{L}_{\text{ALL-TM}}$ , 397
    - $\mathcal{L}_{\text{CFG}}$ , 397
    - $\mathcal{L}_{\text{DFA}}$ , 397
    - $\mathcal{L}_{\text{TM-DECIDER}}$ , 401
    - $\mathcal{L}_{\text{EMPTY-CFG}}$ , 397
    - $\mathcal{L}_{\text{EMPTY-DFA}}$ , 397
    - $\mathcal{L}_{\text{EMPTY-TM}}$ , 397
    - $\mathcal{L}_{\text{EQ-CFG}}$ , 397
    - $\mathcal{L}_{\text{EQ-DFA}}$ , 397
    - $\mathcal{L}_{\text{EQ-TM}}$ , 397
    - $\mathcal{L}_{\text{FINITE-CFG}}$ , 397
    - $\mathcal{L}_{\text{FINITE-DFA}}$ , 397
    - $\mathcal{L}_{\text{FINITE-TM}}$ , 397
    - $\mathcal{L}_{\text{EXP}}$ , 409
    - $\mathcal{L}_{\text{TM-LB}}$ , 401
    - $\mathcal{L}_{\text{TM}}$ , 389
    - $\mathcal{L}_{\text{CFG-00}}$ , 397
    - $\mathcal{L}_{\text{DFA-00}}$ , 397
    - $\mathcal{L}_{\text{TM-00}}$ , 397
    - $\mathcal{L}_{(01)^n}$ , 342, 348, 365
    - $\mathcal{L}_{0^n 1^n}$ , 342, 348
    - $\mathcal{L}_{\text{palindrome}}$ , 347, 348
    - $\mathcal{L}_{\text{repeated}}$ , 342, 348
    - $\mathcal{L}_{\text{unary}}$ , 342
    - $\mathcal{L}_{\text{door}}$ , 340, 341
    - $\mathcal{L}_{\text{HALT}}$ , 392
    - $\mathcal{L}_{\text{pal}}$ , 342, 343
    - $\mathcal{L}_{\text{prime}}$ , 339, 341
    - $\mathcal{L}_{\text{push}}$ , 339, 341, 344, 345, 347
    - $\mathcal{L}_{\text{TM-reject}}$ , 397
  - addition, 359
  - concatenation, 342
  - countable alphabet, 348
  - decidable, 383
    - union, 388
  - distinguisher, 364
  - equality, 355
  - finite description, 348
  - Kleene star, 343
  - non-context free, 373
  - non-recognizable, 395
  - non-regular, 355
  - recognizable, 383
  - recursive, 343
  - reversal, 347, 362–364
  - unary, 411
  - unary, Kleene star, 361
  - uncountable union, 346
  - uncountably many, 348, 385
- $\mathcal{L}(M)$ , language of machine  $M$ , 350
- $\mathcal{L}(M)$ , 365
- Laplace, Pierre-Simon, 211
- LARGE-SUM, 425
- LARGE-SUM, 414

- Las Vegas algorithm, 257  
 last digit,  $3^n$ , 134  
 Latin square, 164  
 law of large numbers, 315  
   strong, 317  
   weak, 317  
 law of total expectation, 283  
 law of total probability, 231–233, 241  
 law of total variance, 323  
 leaf, 108  
 leaping induction, 74  
   postage, 75  
 least common multiple, LCM, 141  
 least-absolute-error, 290  
 least-squares, 290  
 left-matching, 163  
 left-vertex, 162  
 Lehman, Eric, viii  
 Leighton, F. Thomson, viii  
 lemma, 71  
 Let's Make A Deal, 213  
 Levin, Leonid, 418  
 Lewis, Harry, viii  
*Liber Abaci*, 89  
 lifespan, conditioned on age, 283  
 line graph, 174  
 linear growth rate, 117  
 linearity of expectation, 293  
 little-oh, omega, 117–119  
 Littlewood, John, vii  
 Littlewood, John E., 41, 129  
 Lloyd, E. K., 145  
 logarithm, ln, 3  
 logical connector, 26–30  
   rules: associative, commutative,  
   negation, distributive, im-  
   plication, 29  
 Lombard, 230  
 lottery, 288  
 Lovász, Laszlo, viii  
 $\mathcal{L}_{TM}$ , 389  
   undecidable, 390–391  
 Lubell-Yamamoto-Meshalkin, 193, 258  
  
 machine instruction, 344  
 machine-code, 379  
 majority adoption, 11  
 majority voting, 271  
 MAKE-CHANGE, 408, 412  
 making change, greedy algorithm, 79  
 map, coloring, 150  
 mapping reduction, 400  
   prove non-recognizable, 401  
 marginal, 263  
 Marilyn vos Savant (Mach), 213  
 mark and recapture, 276  
 Markov Inequality, 289, 316  
 markup language, 370  
 MATCH, 408  
 matched parentheses, 91, 101–103, 107  
 matching, 161–179  
   augmenting path, 170  
   complete graph, 171  
   greedy, 171  
   jobs on servers, 171  
   network flow, 178  
   perfect, 170  
   residents to hospitals, 172  
   stable, 165  
   students to camps, 171  
   students to courses, 171  
   vertex cover, 172  
   with capacities, 171  
 matching game, 266  
 mathematical induction, *see* induction  
 matrix, 109  
   doubly stochastic, 171  
   Hadamard, 110  
   multiplication, 111  
  
 MAX-FLOW, 408  
 maximal independent set, 175  
 maximal matching, 170  
 maximum chain, 159  
 maximum clique, 429  
 maximum cut, 176  
   probabilistic method, 307  
 maximum flow, 168  
 maximum independent set, 175, 429  
 maximum matching, 170  
 maximum substring sum, 113  
   algorithms, 127–128  
 measurement, 259  
 median, 290  
 medical testing, 231  
   pooling, 302  
 meiosis, 238  
 memory, 377  
 memory, external, 356  
 Menger's Theorem, 157  
 Mersenne prime, 143, 144  
 methane, 92  
 method of differences, 60  
 metric  $k$ -center, 176  
 Meyer, Albert R., viii  
 Micali, Silvio, 197  
 min-cut, 257  
   counting, 258  
 minimum  
   coloring, 175  
   dominating set, 174  
   stretch, 176  
   vertex cover (bipartite), 172  
 model of computing, 9, 377  
 model, epidemics, 7  
 modular arithmetic, 133–138, 141–142  
   exponentiation, 137  
   inverse, 135, 137  
   transitive, 141  
 modular equivalence, 134  
 modulus  
   prime, 135  
 moment generating function, MGF, 289  
 monochromatic, 258  
 Monte-Carlo, 215, 216, 220, 224, 255,  
   257, 271, 277, 278  
   failure, 323  
   variance, 323  
 Monty Hall, 213, 217, 227, 236  
 Moore's law, 123  
 mortgage, 94  
 Motwani, Rajiv, viii  
 $\mu$ , average, 66  
 Mullis, Kary, 197  
 multi-processor tree, 92  
 multigraph, 150, 151, 154  
 multinomial coefficient, 197, 198, 208  
 Multinomial distribution, 274, 276  
 Multinomial Theorem, 198  
 multiparty computation, MPC, 270  
 multiplicative inverse, 135, 137  
 multiplicity rule, 186  
 Myhill-Nerode Theorem, 364  
  
 Nance, Kim, 161  
 Nash, John, 292  
 National Resident Matching, 172  
 natural numbers, 15  
 Nautilus shell, 100  
 negation, 26  
 neighbor, 146  
 nested sum rule, 115–117  
 nett worth versus GDP, 11  
 network  
   collaboration, 147  
   real world, 147  
   road, 147  
   world-wide-web, 147  
 network flow, 168  
   Ford-Fulkerson algorithm, 178  
   maximum matching, 178  
 Nietzsche, Friedrich, 259  
 Nim, 77, 82  
 (NO)-state, 349  
 node, *see* vertex  
 noisy channel, 255, 273  
 non-computable, 334  
   Busy Beaver, 402  
   Kolmogorov complexity, 402  
 non-constructive proof, 203  
 non-identical trials, 276  
 non-polynomial, 409  
 non-recognizable, 395  
    $\mathcal{L}_{TM-DECIDER}$ , 401  
    $\mathcal{L}_{EMPTY-TM}$ , 400  
    $\mathcal{L}_{EQ-TM}$  and  $\overline{\mathcal{L}_{EQ}}$ , 400  
 non-regular, 355, 363  
    $\mathcal{L}_{balanced}$ , 356  
    $\mathcal{L}_{0^n 1^n}$ , 355  
   addition, 359  
   palindromes, 363  
 nondeterminism, 362  
   and complement, 362  
   PDA, 364  
 nondeterministic automaton, 354  
 nondeterministic polynomial, *see* NP  
 NOT/ $\neg$ , 26  
 NP, 413–431  
   3-SAT, 422  
   CIRCUIT-SAT, 418  
   CLIQUE, 415  
   COLORING, 416  
   IND-SET, 423  
   PARTITION, 414  
   nondeterminism, 416  
 NP-complete, 413–431  
   3-COLORING, 428  
   3-SAT, 422  
   BALANCED-BIPARTITE-CLIQUE, 427  
   BIG-CLIQUE, 424  
   BIN-PACKING, 428  
   BOUNDED- $k$ , 426  
   CIRCUIT-SAT, 418  
   CLIQUE, 423  
   DOM-SET, 428  
   FREQ-ITEMS, 423  
   IND-SET, 423  
   KNAPSACK, 428  
   PARTITION, 427  
   SUBSET-SUM, 427  
   VERTEX-COVER, 423  
   X3-SAT, 429  
   coping, 424  
   integer programming, 429  
 $N_{STRONG}$ , 103  
 number theory, 129–145  
   open problems, 144  
  
 $o(\cdot)$ , 117  
 octohedron, 158  
 $O(\cdot)$ , 117–404  
 $\omega(\cdot)$ , 117  
 $\Omega(\cdot)$ , 117  
 one-to-one, *see* 1-to-1  
 1-to-1, 184, 327, 335  
 ONLY IF, 27  
 onto, 184, 327, 335  
 OR/ $\vee$ , 26  
 order notation, *see also* asymptotic  
   analysis, 117–119, 404  
   polynomial, 118  
 Ore's Theorem, 174  
 ork, 101  
 out-degree, 152  
 outcome, 211, 212  
 outcome-probability, 212–217, 233, 260  
 outcome-tree, 212, 213, 215, 233  
   infinite, 232

- P, 403–413  
 2-COLORING, 428  
 2-SAT, 427  
 $P(\cdot)$ , outcome-probability, 216  
 $P = NP$ , 413–431  
 $\mathbb{P}[\cdot]$ , event probability, 217  
 packet transmission, 222  
 palindrome, 91, 104, 107  
   recursive definition, 104  
 Palmer, Thomas H., 281  
 Papadimitriou, Christos, viii  
 paradox  
   barber, 335  
   birthday, 246  
   exchange, 292  
   friends, 155  
   Galton, 221, 235  
   Russell's, 338  
   Simpson's, 239  
   St. Petersburg, 305  
   switching, 292  
 parity, 11, 13  
 parking function, 196  
 parse tree, 92, 369  
 partial fraction, 4  
 partial fractions, 256  
 partial order, 152–153, 159  
   antichain, 152  
   maximum chain, 152  
 partition, 156, 241  
 PARTITION, 427  
 PARTITION, 414  
 partners  
   average, 162  
   total, 162  
 Pascal's Identity, 182, 192  
 Pascal's Triangle, 182  
 Pascal, Blaise, 98, 211  
 password, 289  
 Patashnik, Oren, viii  
 path, 145  
   disjoint, 178  
   Euler, 156  
   length, 145  
   monotonic, 157  
   number of, 156  
   shortest, 145, 157  
   simple, 145  
   stretch, 176  
 PATH, 408, 412, 425  
 PATH-DIR, 425  
 PCP, 393, 399  
   recognizable, 399  
 PDA, 356–357, 371  
   instruction, 357  
   nondeterministic, 364, 372  
 PDF, 260–264  
    $n$  dice, 322  
   build-up, 322  
   conditional, 273  
   dice, 261  
   from CDF, 265, 269  
   joint, 262  
   marginal, 263  
   normalization, 269  
   number of successes, 275  
   plot  
     Binomial, 268  
     uniform, 268  
     waiting time, 268  
   Poisson, 276  
   random walk, 273  
   sum, 270  
   sum to 1, 269  
   waiting time, 268  
   waiting time, sum, 270  
 Pelikán, József, viii  
 perfect matching, 170  
 perfect number, 129, 144  
   even, 143  
 Perlis, Alan, 349  
 permutation matrix, 171  
 permutations, 186  
    $\binom{n}{k_1, \dots, k_r}$ , 197  
   boys and girls, round table, 196  
   random, 223  
   with repetition, 197  
 Petersen graph, 159, 174  
   3-coloring, Hamiltonian path,  
     Euler path, 174  
   not bipartite, 174  
 $\varphi$ , golden ratio, 90, 100  
 Picasso, Pablo, 349  
 Pick's Theorem, 69  
 pigeonhole principle, 202–205, 209  
   born on same day, 205  
   congruence (mod  $k$ ), 205  
   generalization, 209  
   head-hair, 205  
   increasing subsequence, 205  
   prove non-regular, 356  
   social twin, 203  
   subset-sum, 203, 209  
 plain text attack, 136, 143  
 planar graph, 149, 158  
   5-color theorem, 173  
   6-color theorem, 168  
   and polyhedra, 150  
   bound on edges, 150, 168  
   coloring, 167  
   external face, 150  
   integrated circuit design, 149  
   internal face, 150  
   map, coloring, 150  
 planar subgraph, 158  
 Plutarch, 211  
 Poisson distribution, 276  
   variance, 319  
 Poisson-Binomial distribution, 276  
   expected value, 294  
 pokemon, 208  
 poker hands, 187, 207, 219, 222, 237  
 Polya's mouse, 6  
 Polya, George, viii, 78  
 polyhedron, 150, 158  
 polynomial, 403–431  
    $\mathcal{L}_{EXP}$ , 409  
   input format, 411  
   order, 118  
 polynomial, number of coefficients, 196  
 pooling, 302  
 Pope, Alexander, 277  
 Popular Mechanics, 377  
 positive predictive value, PPV, 238  
 Post's Correspondence Problem, *see*  
   also PCP  
 postage, 55, 66, 75  
 power set, 17, 65, 333  
   cardinality, 337  
 PPV, positive predictive value, 238  
 predator-prey game, 292  
 predicate, 31  
 preference list, 164  
 Pregel (river), 151  
 Prentice-Hall, 225  
 Prim's algorithm, 176  
 prime factorization, 75  
 prime number, 75, 129  
   factoring is hard, 134, 137  
   factorization, 133  
   generating, 139  
   infinitely many, 129  
   Mersenne, 143  
   of the form  $3n + 2$ , 142  
   of the form  $4n - 1$ , 142  
   sieve of Eratosthenes, 139  
 probabilistic method, 258, 307  
   3-SAT, 308  
   independent set, 308  
   maximum cut, 307  
   Ramsey numbers, 307  
 probability, 211–216, 325  
   EXCLUSIVE-OR, 223  
   and sets, 216  
   Bayes' Theorem, 233  
   build-up, 241  
   case-by-case analysis, 231  
   complement, 218  
   conditional, 225  
   cumulative, 264  
   device failure, 273  
   dice, 213  
   event, 213  
   failure, 254  
   frequency interpretation, 212  
   genetics, 272  
   guess high or low, 270  
   implication bound, 218  
   inclusion-exclusion, 218, 224  
   intersection, 218  
   joint, 231  
   law of total probability, 231–233  
     more than 2 cases, 241  
   meaning, 211  
   outcome, 212  
   outcome-tree method, 213, 216  
   path, 222  
   sampling bias, 229  
   six steps, 211  
   sum rule, 218, 231  
   survival, 239  
   union, 218  
 probability dist. function, *see* PDF  
 probability function, 216  
 probability space, 216, 221, 223, 227,  
   261, 269  
   infinite, 219  
   uniform, 215, 218  
 product  
    $\prod$ , 3  
   and sums, 125  
   consecutive numbers, 193  
   empty, 4  
   Euler, 144  
   expected value, 297  
 product-state, 353  
 production rule, 365  
 profilactic, 222  
 program translation, 396  
 program verification  
   undecidable, 394  
 proof, 9, 19–22, 41–55  
   Binomial Theorem, 192  
   by contradiction, 21, 47–48  
   by contraposition, 45  
   combinatorial, 193  
   direct proof, 41–44  
   direct with contradiction, 48  
   disproving implication, 44  
   equivalence, 46  
   for all, 43–44, 55–85  
   IF AND ONLY IF, 46  
   induction, 55–85  
   involving GCD, 130  
   involving sets, 48  
   non-constructive, 203  
   picking a template, 48  
   readable, 42  
   recursion, 101–113  
   seven "C's", 50  
   structural induction, 102  
   three steps, 22  
   wrong, 50  
 propane, 92  
 proposition, 25–41  
   compound, 26–30  
   derivation, 29

- quantified, 30
- prosecutor's fallacy, 238
- Prüfer code, 196
- pseudocode, 378
- public key cryptography, 136
- pumping lemma, 364
  - CFG, 375
- punctuation, 364
- pushdown automata, 371
- pushdown automaton, *see* PDA
- puzzles, 11
  - $3n + 1$ , 14
  - 2-contact ebola, 12
  - bipartite birthday, 171
  - break chocolate, 11
  - break the links, 12
  - bridge crossing, 12
  - camel and bananas, 12
  - canibals and pacifists, 159
  - catch the thief, 13
  - chameleon, 12
  - chomp, 12
  - coins in a line, 13
  - comparison scale, 13
  - connection, 6
  - domino combining, 9
  - egg-drop, 12, 98
  - elimination tournament, 11
  - fastest horses, 13
  - find the good processor, 13
  - fly around the world, 13
  - fox, chicken and corn, 12
  - graph path game, 171
  - guess the number, 13
  - jugs, 140
  - jugs, Die Hard, 131
  - matching subset sums, 14
  - Monty Hall, 213
  - non-attacking queens, 159
  - pardon the prisoners, 13
  - pirates, 11
  - poisoned wine, 13
  - quarters game, 11
  - replacing cards, 13
  - security guards, 12
  - sliding tiles, 83
  - Sudoku, 173
  - tiling, 12
  - trash compactor, 99
  - trucks delivering payload, 12
  - voyage around the world, 174
  - with dominos, 12
- Pythagoras, 58
- Pythagoras' Theorem, 143
- Pythagorean Triple, 143
  - primitive, 143
- $\mathbb{Q}$ , countable, 331, 336
- quantifier, 30–41
  - ALL/ $\forall$ , 31
  - combining with connectors, 32
  - existential, 32
  - EXISTS/ $\exists$ , 32
  - mixing, 33
  - negation, 33
  - proofs, 34
  - universal, 32
- Quicksort, expected runtime, 286
- quine, 399
- quotient, 129
- quotient remainder theorem, 24, 69, 84, 129, 139
- Rademacher, Paley & Zygmund, 256
- Radhakrishnan, 258
- radio frequencies, 166
  - assignment, 177
- radix, 411
- Radó, Tibor, 402
- RAM, 373, 377
- Ramanujan, Srinivasa, 129
- Ramsey number, 24, 157
  - lower bound, 307
- random, 212
- random graph, 273, 276
- random variable, 259–277
  - Bernoulli, 265
  - binary, 265
  - Binomial, 273
  - conditional probability, 260
  - event, 260
  - function of, 272
  - independent, 263, 269, 272
  - indicator, 259, 264, 265, 269
  - match, 273
  - sum, 270
  - transformation, 272
  - uniform, 265
- random walk, 223, 250–252, 276
  - expected hitting time, 290
  - PDF, 273
  - variable step size, 322
- randomization device, 212
- randomized algorithm, 256–257
- randomness of string, 402
- rational numbers, 16
  - countable, 336
- RBT, *see also* tree, 92, 105, 106, 108
  - counting, 112
- read-write head, 378
- real numbers, 15
  - uncountable, 332
- recognizable, 383
  - $\mathcal{L}_{\text{EQ-CFG}}$ , 398
  - $\mathcal{L}_{\text{TM}}$ , 389
  - closure, 400
  - PCP, 399
- record breaking, 300, 305
- recurrence, 89–90, 113
  - $k$ th order linear, 95
  - analysis with linear algebra, 97
  - exponential sequence, 95
  - gambler's ruin, 251, 256
  - generating function, 93, 111
  - integration bound, 121
  - linear, 93, 95
  - unfold, 121
- recursion, 85–101, 113
  - $\mathcal{L}_{\text{pal}}$ , 343
  - $\binom{n}{k}$ , 182
  - Akra-Bazzi formula, 96
  - asymptotic analysis, 124
  - Catalan, 98
  - CFG, 365
  - change of variable, 96
  - checklist, 90
  - computing determinant, 98
  - derangement, 201
  - on trees, 106
  - Pascal's recursion, 98
  - proof, 101–113
  - proof of correctness, 87
  - Towers of Hanoi, 99
  - unfolding, 87
- recursion vs. induction, 86
- recursive case, 86
- recursive function, 86
- recursive program, 90
  - correctness, 90
  - runtime recurrence, 90
- recursive progress, 86
- recursive set, 91
  - $\mathbb{N}$ , 91
  - basis, 91
  - constructor, 91
  - finite binary strings, 91
  - matched parentheses, 91, 101–103
- minimality, 91
- multiples of 3, 108
- palindromes, 91
- powers of 3, 91
- recursive structures
  - derivation, 93
  - rooted binary tree, 92
  - rooted full binary tree, 93
  - trees, 92, 158
- reductio ad absurdum*, 21, 47–48
- reduction, 391–392, 400
  - 3-SAT to 3-COLORING, 428
  - 3-SAT to IND-SET, 422
  - 3-SAT to INT-PROGRAM, 429
  - 3-SAT to SUBSET-SUM, 427
  - 3-SAT to solitaire, 429
  - CIRCUIT-SAT to 3-SAT, 422
  - CLIQUE to BIG-CLIQUE, 424
  - CLIQUE to CIRCUIT-SAT, 419
  - CLIQUE to FREQ-ITEMS, 424
  - COLORING to scheduling, 429
  - PARTITION to BIN-PACKING, 428
  - PARTITION to KNAPSACK, 428
  - VERTEX-COVER to DOM-SET, 428
- regret, 166, 171
- regular, 154
- regular expression, 342, 346, 360
  - and complement, 343
  - DFA, 355
  - intersection and complement, 361
- regular graph, 148, 170
- regular language, 351
  - closure, 363
  - complement, 352
  - concatenation, 353
  - intersection, 352
  - Kleene star, 353
  - union, 352
- pumping lemma, 364
- relation
  - binary, 152
  - reflexive, 152
  - transitive, 152
- relatively prime, 130, 139, 202
- RELPRIME, 412
- remainder, 129
- representation, powers of 2, 77, 81
- residual graph, 164
- resource allocation
  - and Sperner's Lemma, 160
  - envy-free, 160
- restricted choice, 224, 239
- return time, 276
- reversal, 347, 362
  - language, 347, 364
- RFBT, *see also* tree, 93, 108
- RFTT, 108
- Rice's Theorem, 399
- Riemann Hypothesis, 144
- right-vertex, 162
- Ringel, 160
- Rivest, Ron, 136
- rock, paper, scissors, 270
- rooted binary tree, 105
- Rosa, 160
- Rota, Gian-Carlo, 327
- roulette, 252, 255, 277
  - Monte Carlo, 240
- rounding operation, 83
- RSA, vii, 129, 136–138
  - algorithm, 137
  - mathematics of, 138
- RT, 108
- RTT, 108
- runtime, 114
- runtime function
  - compare, 118
- Russell's Paradox, 338
- Russell, Bertrand, 113, 277, 309, 338



- Russian roulette, 222  
Rutherford, Ernst, 403
- sample average, 278  
sample space, 216, 221  
sampling bias, 229  
    we're growing taller, 234  
sampling without replacement, 305, 320  
satisfiability, 418–421, 426  
    2-SAT is in P, 427  
    k-SAT, 425  
Satterwhite, Valery, 259  
scheduling, 166, 178, 429  
    NP-complete, 429  
Schlicler, Dean, 7  
scratch-paper, 349  
search, 249  
    binary, 249  
    sorting approach, 249  
search tree, 92  
self reference, 85  
self-replicating program, 399  
sequence, 17  
sequences with repetition, 197  
Sesame Street, 179  
set, 15–17  
    countable, *see* countable  
    difference, 53  
    infinite, 327–339  
    measure zero, 338  
    operations, 16  
        associative, 17  
        commutative, 17  
        complement, 16  
        distributive, 17  
        intersection, 16  
        union, 16  
    prove equality, 48  
    recursive, 91  
    ugly, 337  
seven bridges of Königsberg, 151  
sex in America, 161  
Shakespeare, 165  
Shamir, Adi, 136  
Shank's transform, 127  
Shaw, George Bernard, 243  
Sherlock Holmes, 225, 293  
Shimura, Goro, 101  
SHORTEST-PATH, 408  
sieve of Eratosthenes, 139  
 $\Sigma$ , alphabet, 341  
 $\Sigma$ , sum, 113–129  
 $\sigma$ , *see* standard deviation  
 $\sigma^2$ , *see* variance  
Simpson's paradox, 239  
simulate dice, 304  
simulation, 215  
simulation theorem, 408  
Sisyphus, 10  
six-shooter, 222  
size(), 106, 108  
social golfer problem, 11  
social network, 8  
    marketing, 11  
Social Organization of Sex., 161, 162  
social twin, 203, 248  
solitaire, NP-complete, 429  
solvable, 413  
sorter, 425  
spam, 236  
spanning  
    minimum weight, 176  
spanning tree, 158, 168  
    low stretch, 176  
        complete graph, 176  
speed dating, 8, 11  
Spencer, 258  
Sperner's Lemma, 160  
    Brouwer Fixed Point Thm., 160  
    envy-free resource allocation, 160  
spiral  
    galaxy, 100  
    golden, 100  
    logarithmic, 100  
    storm, 100  
spread, 320  
Srinivasan, 258  
St. Petersburg Paradox, 305  
stable marriage, 164, 171  
    all matchings, 171  
    proposer wins, 166, 172  
    volatile pair, 165  
stable matching, *see* stable marriage  
standard deviation, 311, 320  
Stanley, Richard P., viii  
star, 147  
state  
    halting,  , 377  
    product, 353  
    subset, 354  
Stirling number, 191  
Stirling's formula, 120, 121, 210  
Stirzaker, David, viii  
stochastic matrix, 171  
stretch, 176  
string  
    concatenation, 104  
    reversal, 104  
string patterns, 342  
strong induction, 70, 75, 76  
    Fund. Theorem of Arithmetic, 75  
    games, 77  
    making change, 79  
    representation, powers of 2, 77  
strong law of large numbers, 317  
strongly connected, 152  
structural induction, 101–113  
    and strong induction, 103  
    checklist, 106  
    on  $\mathbb{N}$ , 103  
    proof of, 112  
    tree properties, 105–106  
subdivision, 160  
subgraph, 157  
subgraph, induced, 157  
subsequence, monotonic, 159  
subset relation, 16  
subset-state, 354, 355  
SUBSET-SUM, 415, 425  
    build-up method (dynamic programming), 427  
    reduction to PARTITION, 427  
substitution rule, 365  
Sudoku, 173  
sum, 113–129  
     $\Sigma$ , 3, 113  
    addition rule, 114  
    and Taylor series, 127  
    arithmetic, 59  
    bounding terms, 125  
    common sums, 115  
    constant rule, 114  
    cubes, 61  
    dice, 294  
    divergent, 127  
    empty, 4  
    expected value, 293  
    geometric, 56, 59, 99  
        integration bound, 121  
    integer powers, 58, 70, 120  
    integers, 58  
    maximum substring, 113  
    nested sum rule, 115–117  
    odd numbers, 61  
    reciprocal of squares, 126  
    rules, 114–117  
    squares, 60  
    telescoping, 70, 126, 127  
    using derivative, 291  
    using integration, 119–129  
    variance, 313  
sum rule, disjoint sets, 179  
sum rule, probability, 231  
sum share, 270  
summand, 113, 115, 119  
summation, 113–129  
    accelerate, 127  
    and differentiation, 126  
    by parts, 126  
    index, 113  
superpermutation, 195  
surjection, 188, 327, 335  
survey design, 178  
survival probabilities, 238  
switch or not, 213
- Taniyama, 101  
tape  
    beacon symbol, \*, 377  
    blank symbol,  $\sqcup$ , 377  
    erase, 394  
Tardos, Éva, viii  
tautology, 36  
telescoping sum, 70, 126, 127  
ternary strings, 319, 336  
Texas Holdem, 238  
theorem, 19  
 $\Theta(\cdot)$ , 117–404  
tiling  
    domino, 12, 83  
    triomino, 72, 81, 82  
time complexity, 403–413  
tinkering, 7, 10, 60  
TM, *see* Turing Machine  
Tolstoy, Leo, 101  
topological sort, 159  
total expectation, 283, 295  
    dice rolls, 285  
    different types of coins, 284  
    waiting time, 285  
total order, 152  
total probability, 231–233, 241  
totient function  $\phi$ , 142  
tournament, 68, 152, 223, 258  
    acyclic, 159  
    Hamiltonian path, 174  
    inconsistency, 68  
    ranking, 68  
    ranking, strong induction, 81  
    round-robin, 68  
    top-dog, 153  
Towers of Hanoi, 99  
TQBF, 430  
trace, 350  
transcendental number, 337  
transducer, 359  
    Turing Machine, 381  
transfinite realm, 333  
transposed conditional, 230  
traveling salesperson, *see* TSP  
tree, 92–93, 149  
    2-colorable, 168  
    BFS, 158  
    binary, 108  
    binary, full, 108  
    child node, 92  
    child subtree, 92  
    coloring, 167  
    connected acyclic graph, 173  
    graceful labeling, 160  
    greedy coloring, 173  
    height(), 106, 108  
    induction, 173  
    Kraft inequality, 109  
    merge, 109  
    ordered, 92

- properties, 93
- recursive definition, 92, 105
- recursive function on, 106
- root, 92
- rooted, 108
- rooted binary, 92, 105–106
- rooted full binary, 93
- size(), 106, 108
- spanning, 158
  - minimum weight, 176
- spanning tree
  - minimum, 176
- ternary, 108
- ternary, full, 108
- true quantified boolean formula, 430
- Truman, Harry, 229
- truth table, 28–30
- TSP, 177, 403, 411
- Tukey, John, 113
- Turán, 175
- Turing Award, 104, 325
- Turing Machine, 373, 377–387
  - \*01\*, 381
  - $\{0^{*n}1^{*n}\}$ , 403
    - using two tapes, 405
  - and circuits, 418, 426
  - copying, 383
  - counter, 394
  - decider, 383
  - encoding, 384
  - fast, 405
  - finite binary string, 385
  - infinite loop, 383
  - linearly bounded, 401
  - LR vs. LRS, 386
  - machine-instruction, 378
  - multiplication, 380
  - programmable, 388
  - read only, 386
  - recognizer, 383
  - reversal, 383
  - sketch, 387–388
  - squaring, 383
  - transducer, 381
  - two-tape using one, 406
  - two-tapes using one, 388
  - universal, 389
  - $\{w#w\}$ , 377, 379
  - $\{ww\}$ , 381
- Turing, Alan, 136, 325, 334, 377, 390
- Twain, Mark, vii, 41, 85
- twin-prime, 346
- 2-hop neighbor, 178
  
- ugly set, 337
- Ullman, Jeffrey, viii
- ultimate-debugger, 387, 396
  - does not exist, 392
- uncountable, 361
  - continuum, 333
  - infinite binary strings, 332
  - measure zero set, 338
- real numbers, 332
- undecidable, 387–403
  - $\mathcal{L}_{\text{ALL-CFG}}$ , 398
  - $\mathcal{L}_{\text{ALL-TM}}$ , 399
  - $\mathcal{L}_{\text{TM-DECIDER}}$ , 401
  - $\mathcal{L}_{\text{EMPTY-TM}}$ , 398
  - $\mathcal{L}_{\text{EQ-CFG}}$ , 398
  - $\mathcal{L}_{\text{EQ-TM}}$ , 399
  - $\mathcal{L}_{\text{FINITE-TM}}$ , 399
  - $\mathcal{L}_{\text{TM-LB}}$ , 401
  - $\mathcal{L}_{\text{TM}}$ , 390–391
  - $\mathcal{L}_{\text{TM-OO}}$ , 398
  - PCP, 393
  - program verification, 394
  - prove, 392
  - TM language is regular, 399
- uniform probability space, 215, 218
- uniform random variable, 265
- union, countable, 330
- universal set, 150
- universal Turing Machine, 389
- $U_{\text{TM}}$ , 389
- Unix, 342
- unordered pair, 146
- unsolvable, 387–403
- urn, 222, 224
- utility graph, 149
  
- value(), 104–105
- van Lint, J. H., viii
- Vandermonde convolution, 192, 208
- Vardi, Ilan, 56
- variance, 309–318
  - 3-sigma rule, 315
  - Bernoulli, 312
  - Binomial, 312
  - computing, 311
  - conditional, 323
  - iterated, 323
  - maximize, 322
  - non-negative, 312
  - Poisson, 319
  - reduce, 320
  - sum, 313–315
  - uniform, 312
  - waiting time, 312
- Venn diagram, 17, 48, 53, 200
- Verghese, Abraham, 129
- verifiable, 413
  - and nondeterminism, 417
  - polynomially, 414
- verification, hello world, 14
- vertex, 145
- vertex cover, 168, 173, 174
  - and independent set, 175
  - matching, 172
- VERTEX-COVER, 423, 425
- Vesztegombi, Katalin, viii
- viral marketing, 174
- virus, 400
- volatile pair, 165
- voltage fluctuations, 322
  
- von Neumann, John, 7, 25, 431
- voting, 271
  
- waiting time, 285, 291
  - $k$  successes and  $\ell$  failures, 275
  - $r$  successes, 274, 291
  - boy, 268
  - boy and girl, 269, 286
  - equal boys and girls, 291
  - expected, 294, 305
  - geometric distribution, 268
  - memoryless, 274
  - packet transmission, 268
  - squared, 286, 294
  - string of flips, 269
  - to success, 268
  - two boys, 268, 285
- Wald, Abraham, 230
- Ward, Artemus, 55
- water-bullet-roulette, 222
- Watson, Thomas J., 431
- weak law of large numbers, 317
- web-graph, 147
- weighted degree, 156
- weighted graph, 150
  - Handshaking Theorem, 151
- well-ordering principle, 20–22
  - induction, 63, 70, 338
- West, Douglas B., viii
- wheel, 147
- WHENEVER, 27
- Wierstrauss, 126
- wildcard symbol, 342, 347
  - vs. Kleene star, 346
- Wiles, Andrew, 53
- Wilf, Herbert, 55
- Williams, Serena, 3
- Wilson, R. J., 145
- Wilson, R. M., viii
- Wilson, Sariah, 161
- witness, 414
- witness reliability, 237
- world-wide-web, 147
- worst case analysis, 404
- Wozniak, Steve, 149
- WWII, 133, 136, 230
  
- x3-SAT, 429
- XML, 371
- XOR, 223
  
- $\overline{\text{YES}}$ -set, 341–350
- $\overline{\text{YES}}$ -state, 349
- younger-twin-prime, 346
  
- $z$ -score, 317
  - mean and variance, 319
- Zeit, Paul, viii
- Zermelo-Fraenkel with Choice, ZFC, 15, 333, 338
- zero knowledge proof, 430